Vascular Fluid Structure Simulation

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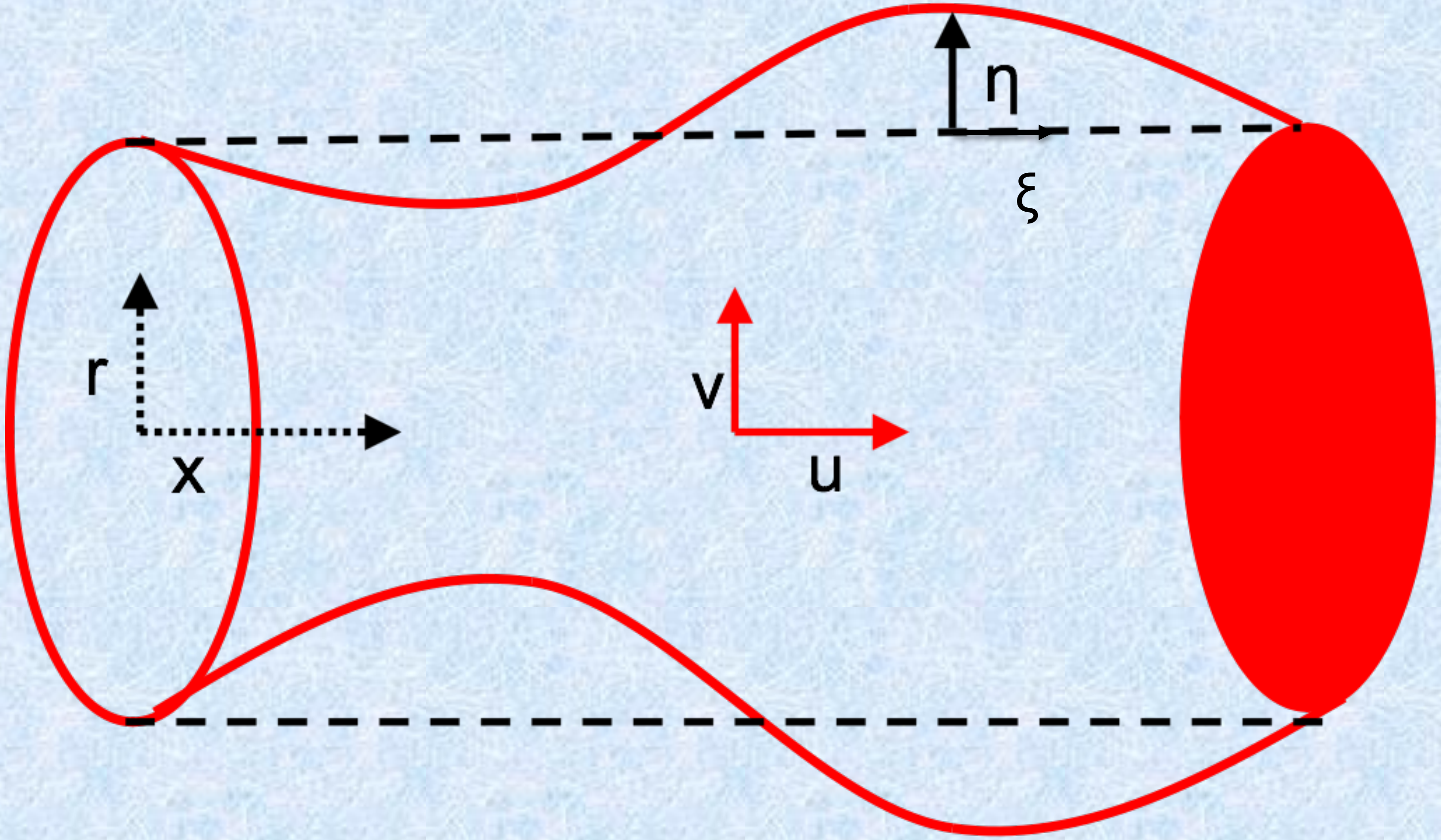
Fluid-Structure Interactions

- Goal:

  Utilize a set of programs to simulate the blood flow in artery
- Two components
  - Fluid (blood) modeled by Navier-Stokes equations on a 2D mesh
  - Solid structure (vessel wall) modeled by two partial differential equations giving radial and longitudinal deformation of wall from its resting state
- Solve two components separately
- Blood flow causes deformation of the vessel wall and deformation of the wall changes the boundary conditions of blood flow.

Artery model



Quarteroni, Alfio; Tuveri, Massimiliano; Veneziani, Alessandro. "Computational vascular fluid dynamics: problems, models, and methods" Comput Visual Sci 2:163-197 (2000).

Olufsen Equations

Fluid Equations(INS)

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \upsilon\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} - \frac{u^2}{r^2}\right)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial x^2}\right)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial x}$$

Boundary Conditions

$$[u]_{r=a} = \frac{\partial \eta}{\partial t} \quad \text{and} \quad [w]_{r=a} = \frac{\partial \xi}{\partial t}$$

Olufsen Equations

Vessel Wall Equations

$$M_0 \frac{\partial^2 \xi}{\partial t^2} + L_x \frac{\partial \xi}{\partial t} + K_x \xi$$

$$= \frac{E_x h}{1 - \sigma_\theta \sigma_x} \frac{\partial^2 \xi}{\partial x^2} + \left(\frac{T_{t_0} - T_{\theta_0}}{a} - \frac{E_x h \sigma_x}{a(1 - \sigma_\theta \sigma_x)} \right) \frac{\partial \eta}{\partial x} - \mu \left[\frac{\partial w}{\partial r} + \frac{\partial u}{\partial x} \right]_a$$

$$M_0 \frac{\partial^2 \eta}{\partial t^2} + L_r \frac{\partial \eta}{\partial t} + K_r \eta$$

$$= T_{t_0} \frac{\partial^2 \eta}{\partial x^2} + \left(\frac{T_{\theta_0}}{a^2} - \frac{E_\theta h}{a^2 (1 - \sigma_\theta \sigma_x)} \right) \eta + \frac{E_\theta h \sigma_\theta}{a(1 - \sigma_\theta \sigma_x)} \frac{\partial \xi}{\partial x} + \left[p - 2\mu \frac{\partial u}{\partial r} \right]_a$$

Algorithm

1. Solve Navier-Stokes equations(INS) for blood velocity(u,w) and pressure(p) on a 2D mesh
2. Solve structure equations for radial and longitudinal deformations(η,ξ) of vessel wall on a 1D mesh
3. Update mesh using η , ξ , since vessel wall has moved
4. Update radial velocity at vessel wall, since radial blood velocity at vessel wall must equal radial wall velocity
5. Repeat Step 1-4 until a stable solution is reached
6. $t = t + \Delta t$
7. Continue from Step 1

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$$0 = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial x}$$

Navier-Stokes equations

To solve INS:

Use continuous Galerkin finite element method to approximate the equations

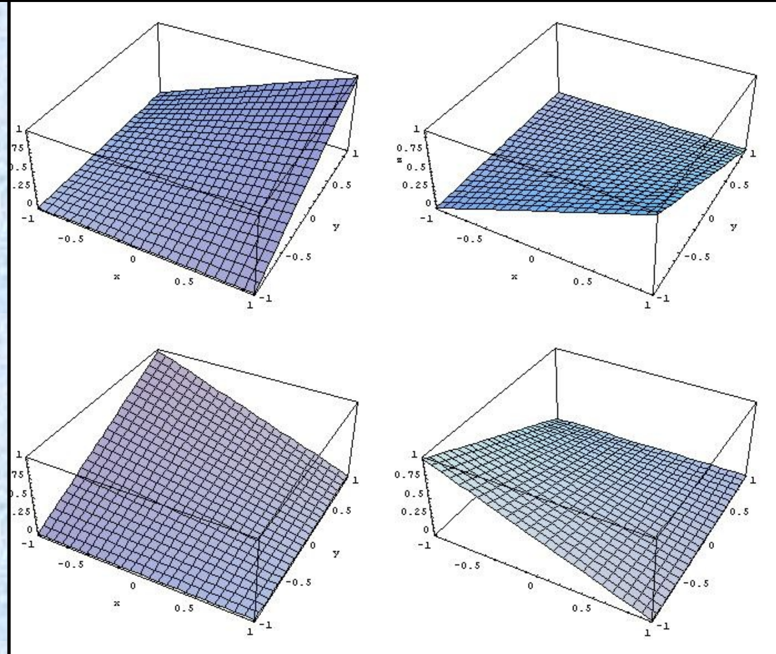
$$[M^e]\{DU^e\} + [C^e]\{P^e\} + [K^e]\{U^e\} = 0$$

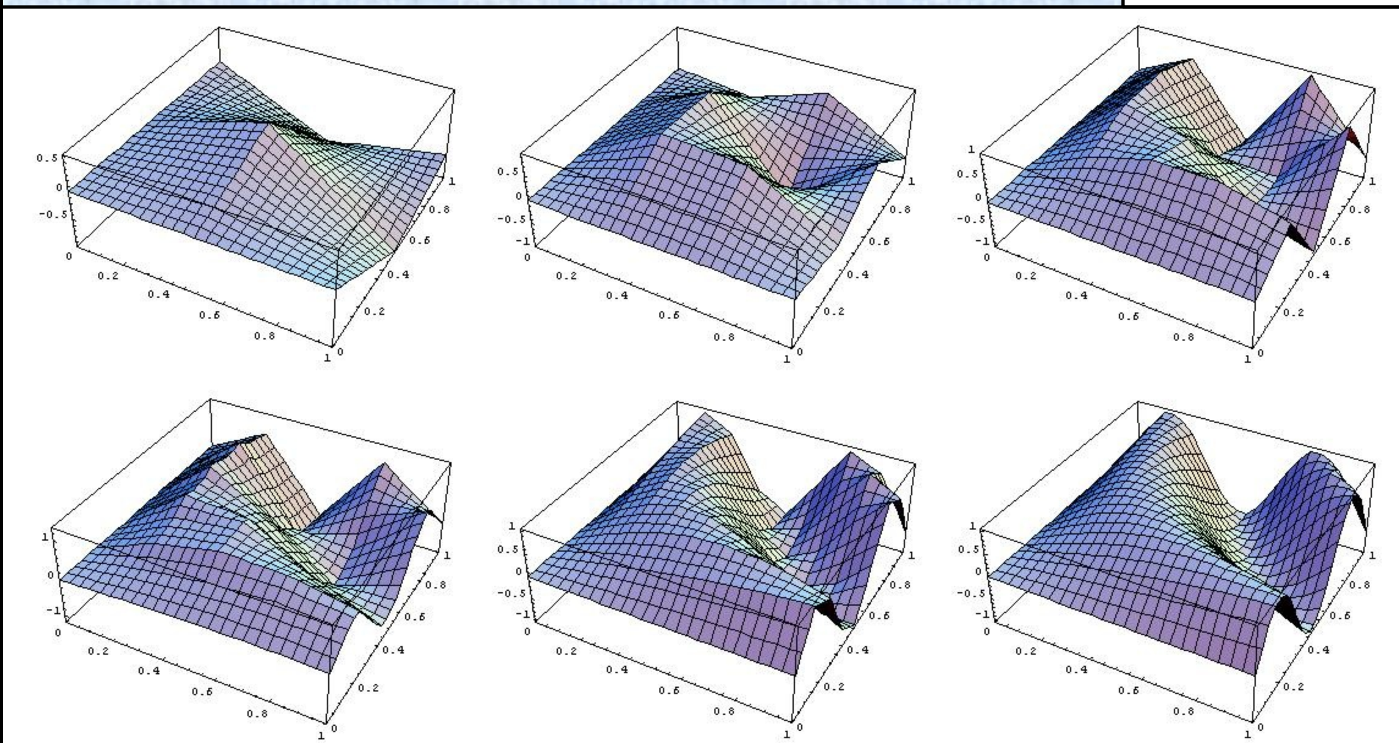
$$[M^e]\{DW^e\} + [C^e]\{P^e\} + [K^e]\{W^e\} = 0$$

$$[M^e]\{U^e\} + [K^e]\{W^e\} = 0$$

Finite elements

- Divide domain into parts
- Seek approximate solution over each part
- Assemble the parts





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Structure Equations

To solve Structure Equations :

1. Use continuous Galerkin finite element method

$$\{M^e\}\frac{\partial^2}{\partial t^2}\{X^e\}+\{C^e\}\frac{\partial}{\partial t}\{X^e\}+\{K^e\}\{X^e\}+\{D^e\}\{N^e\}=\{Q^e\}+\{S^e\}$$

3. Use Newmark method to solve system of second order PDE

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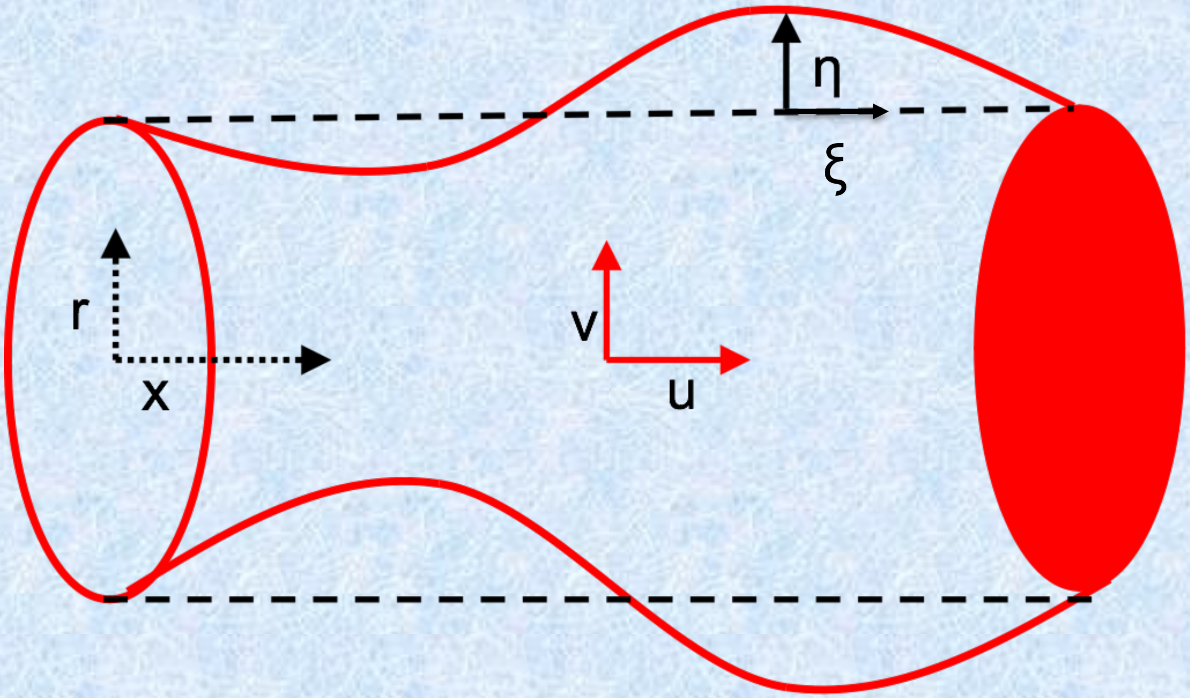

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Plans For Future

- After the program for 2D INS equation can run , Artery will be modeled in 3 dimensional instead of 2D and the blood vessel will be in 2D structure

Acknowledgements

- Mentors: Kwai Wong

Reference:

[1] : Quarteroni, Alfio; Tuveri, Massimiliano; Veneziani, Alessandro. “Computational vascular fluid dynamics: problems, models, and methods" Comput Visual Sci 2:163-197 (2000).

[2] : *Johnny T. Ottesen, Mette S. Olufsen, Jesper K. Larsen, “Applied Mathematical Models in Human Physiology (Siam Monographs on Mathematical Modeling and Computation).“ SIAM, 2004.